ON INTERSECTION OF SIMPLY CONNECTED SETS IN THE PLANE

E. D. TYMCHATYN AND VESKO VALOV

Abstract. Several authors [2] and [7] have recently attempted to show that the intersection of three simply connected subcontinua of the plane is simply connected provided it is non-empty and the intersection of each two of the continua is path connected. In this note we give a very short complete proof of this fact. We also confirm a related conjecture of Karimov and Repovš [7].

1. Introduction

A homology (resp., singular) cell is a compact metric space whose Vietoris (resp., singular) homology groups are trivial. Helly [6] proved the following result which is now known as the Topological Helly Theorem:

Theorem 1.1. *Let $\mathcal{S} = \{S_0, ..., S_m\}$, $m \geq n$, be a finite family of homology cells in $\mathbb{R}^n$ such that the intersection of every subfamily $\mathcal{H}$ of $\mathcal{S}$ is nonempty if the cardinality $|\mathcal{H}| \leq n+1$ and it is a homology cell if $|\mathcal{H}| \leq n$. Then $\cap_{i=0}^{i=m} S_i$ is a homology cell.*

Versions of Theorem 1.1 for singular homology have been proved by Debrunner [5] and Alexandroff and Hopf [1, p. 295] for open sets in $\mathbb{R}^n$ and simplicial complexes in $\mathbb{R}^n$, respectively.

A topological space is said to be simply connected if it is path connected and has trivial fundamental group. It is known [4] that a compact subspace of the plane is a singular cell if and only if it is simply connected.

In section 2 of the paper [6] Helly proved that if S_i, $i = 1, .., 4$, are singular cells in $\mathbb{R}^2$ such that all intersections $S_{i_1} \cap S_{i_2} \cap S_{i_3}$ are singular cells, then $\cap_{i=1}^{i=4} S_i$ is not empty. Hence to prove the Topological Helly Theorem for singular cells in $\mathbb{R}^2$, it suffices to prove the following:

Proposition 1.2. *Let S_0, S_1 and S_2 be three simply connected compacta in the plane such that the intersection of any two of them is path connected and $\cap_{i=0}^{i=2} S_i \neq \emptyset$. Then $\cap_{i=0}^{i=2} S_i$ is simply connected.*

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Bogatyi [2] has pointed out that no complete proof of this proposition can be found in the literature. He proved the proposition in the special case that S_i are Peano continua. Karimov and Repovš [7], established that, with the hypotheses of Proposition 1.2, $\cap_{i=0}^{i=2}S_i$ is cell-like connected (i.e., every two points can be connected by a cell-like continuum). We prove Proposition 1.2 by showing that $\cap_{i=0}^{i=2}S_i$ is path connected. We also give an affirmative answer to a conjecture of Karimov and Repovš [7] by proving the following proposition:

Proposition 1.3. *If X and Y are compact AR's in the plane, then so is each component of $X\cap Y$.*

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2. Proof of Proposition 1.2

Since the intersection of any family of simply connected sets in the plane has a trivial fundamental group with respect to each of its points, it suffices to show that $\displaystyle\bigcap_{i=0}^{i=2}S_i$ is path connected. Suppose this is not the case and let 0 and 1 be two points in distinct arc components of $\displaystyle\bigcap_{i=0}^{i=2}S_i$. Let $I\subset S_0\cap S_1$, $J\subset S_0\cap S_2$ and $K\subset S_1\cap S_2$ be arcs from 0 to 1. Consider the components J_n, $n=1,2,..$, of $J\backslash\big(I\cup K\big)$ which are not in S_1. Since 0 and 1 are end-points of J it follows that no J_i separates $I\cup J\cup K$. Also, no J_i lies in a bounded component of $\mathbb{R}^2\backslash\big(I\cup K\big)$ because if the locally connected continuum $I\cup K$ separates J_i from ∞ in $\mathbb{R}^2$, then some simple closed curve in $I\cup K\subset S_1$ would do so as well. Since S_1 is simply connected, this would imply $J_i\subset S_1$.

We are going to construct for every $n\geq 1$ an arc $J^n\subset S_0\cap S_1$ in $I\cup J\cup K$ from 0 to 1 such that $J^n\cap\big(J_1\cup..\cup J_n\big)=\emptyset$. Let D_1 be the bounded component of $\mathbb{R}^2\backslash(I\cup J\cup K)$ whose boundary contains J_1 and $J_1\cup I\cup K$ separates D_1 from infinity in $\mathbb{R}^2$. Then $D_1\subset D_I\cap D_K$, where D_I (resp., D_K) is the component of $\mathbb{R}^2\backslash(I\cup J)$ (resp., $\mathbb{R}^2\backslash(J\cup K)$) containing D_1. Then D_I and D_K are bounded because J_1 does not separate $I\cup J\cup K$. Note that $D_I\subset S_0$ and $D_K\subset S_2$. Let D be the component of $D_I\cap D_K$ containing D_1. Then $D\subset S_0\cap S_2$ and $J_1\subset\overline{D}\subset S_0\cap S_2$. Moreover, $Fr(D)\subset I\cup J\cup K$. It is well known [9] that each continuum contained in the union of finitely many arcs is rim-finite and, hence, locally connected. So $Fr(D)$ is locally connected. Let $C\subset Fr(D)$ be the simple closed curve that separates D from ∞ in $\mathbb{R}^2$. Then there is an arc $J^1\subset\big(J\cup C\big)\backslash J_1\subset S_0\cap S_2$ from 0 to 1. Obviously, $J^1\subset\mathbb{R}^2\backslash J_1$ since $J_1\subset D$. Suppose we already constructed an arc $J^n\subset S_0\cap S_1$ in $I\cup J\cup K$

from 0 to 1 such that $J^n \cap \big(J_1 \cup .. \cup J_n\big) = \emptyset$. If $J^n \cap J_{n+1} = \emptyset$, let $J^{n+1} = J^n$. If $J^n \cap J_{n+1} \neq \emptyset$, we repeat the above arguments with J^n in place of J and J_{n+1} in place of J_1 to obtain an arc $J^{n+1} \subset S_0 \cap S_2 \cap \big(J^n \cup I \cup J \backslash J_{n+1}\big)$ from 0 to 1. By induction, we construct a sequence of arcs $\{J^n\}_{n=1}^{\infty}$ from 0 to 1 with $J^{n+1} \subset S_0 \cap S_1 \cap \big(I \cup J \cup K \backslash \bigcup_{i=1}^{n+1} J_i\big)$. Let $J^* = \limsup J^n$. Then $J^* \subset \big(S_0 \cap S_2\big) \cap \big(I \cup J \cup K \backslash \bigcup_{i=1}^{\infty} J_i\big) \subset S_1$ is a continuum from 0 to 1. As above, J^* is locally connected. So, there is an arc in J^* from 0 to 1 which contradicts the fact that 0 and 1 are in distinct arc components of $\bigcap_{i=0}^{i=2} S_i$.

3. Proof of Proposition 1.3

Let C be a component of $X \cap Y$. If K is the topological hull of C, then $K \subset X$ and $K \subset Y$ since neither X nor Y separates $\mathbb{R}^2$. So, $K = C$. By unicoherence of $\mathbb{R}^2$ it follows that $Fr(C)$, the boundary of C in $\mathbb{R}^2$, is connected.

By the well-known result of Borsuk [3] (that every locally connected plane continuum not separating the plane is an AR), it remains to prove that C is locally connected. Since C is a continuum in the plane, it suffices to prove that $Fr(C)$ is locally connected. To prove this it suffices to show that every pair of points of $Fr(C)$ is separated by a finite set (see [10, p. 99]).

Since X is simply connected, locally connected subcontinuum in the plane, by [10, ch. IV], all true cyclic elements of X are topological disks D_i such that the cardinality of $D_i \cap D_j$ is at most 1 for $i \neq j$ and, if the sequence $\{D_i\}$ is infinite, then $\lim diam D_i = 0$. Hence, each $Fr(D_i)$ is a simple closed curve and $Fr(X) = X \backslash \bigcup int(D_i)$ is a locally connected continuum with a particularly simple structure. Let x and y be distinct points in $Fr(C) \subset Fr(X) \cup Fr(Y)$. If x and y do not both lie in any one cyclic element of X, then an one point set separates x and y in X and, hence, in C. Thus, we may suppose that there are cyclic elements D in X and E in Y with $x, y \in D \cap E$. Now x in $int(D)$ implies there is a neighborhood W of x in $Fr(X) \cup Fr(Y)$ with $\overline{W} \subset int(D)$. Then a finite set P separates $Fr(Y) \backslash W$ from x in $Fr(Y)$ since $Fr(Y)$ is rim-finite. Hence, P separates x from $Fr(X) \cup Fr(Y) \backslash W$. So we may suppose $x, y \in Fr(D) \cap Fr(E)$ (see [8, 49.V, Theorem 3, p. 244]).

Let F be a two-point set in $Fr(E)$ which separates x and y in $Fr(E)$. Then F separates x and y in $Fr(Y)$ [10, IV.3.1, p. 67]. Since D is hereditary normal, there is a closed set $A \subset D$ which separates x and y in D and such that $A \cap Y \subset F$. Since D is unicoherent, a component A' of A separates x and y in D. It is now a routine exercise to construct an arc $A'' \subset D$ such that A'' separates x and y in D and $A'' \cap Y \subset F$. If we also take A'' to be irreducible

with respect to separating x and y in D (see [8, V.49, Theorem 3, p.244]), then $A'' \cap Fr(D)$ will contain just two points c and d. As above, A'' separates x and y in X because D is a cyclic element of X. So $A'' \cap \big(Fr(X) \cup Fr(Y)\big) \subset F \cup \{c, d\}$ separates x and y in $Fr(C) \subset \big(Fr(X) \cup Fr(Y)\big) \subset X$ .So, $Fr(C)$ is rim-finite, hence, locally connected.

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Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: tymchat@math.usask.ca

Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada
E-mail address: veskov@nipissingu.ca